PHYS5150 — PLASMA PHYSICS

LECTURE 14 - MAGNETOHYDRODYNAMICS

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Spring 2023

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1 EXAMPLE: THE LANGMUIR-CHILD LAW

As an example we consider the emission of an electron beam from a filament in a vacuum tube. Between the two electrodes an electric field E is applied. Now, at the filament

$$\text{emitted electron current:}\, j = n\,q\,u = \text{constant, i.e.}\nabla n\mathbf{u} = 0.$$

$$\text{emission velocity: } u_0$$

$$\text{emission energy: } \frac{m}{2}u_0^2$$

momentum equation:

$$nmu\frac{\mathrm{d}u}{\mathrm{d}x} = nqE = -neE$$

$$\frac{m}{2}\frac{\mathrm{d}u^2}{\mathrm{d}x} = -e\frac{\mathrm{d}\phi}{\mathrm{d}dx}$$

$$\frac{mu^2}{2} - e\phi = \text{constant} = \frac{mu_0^2}{2}$$

$$u^2 = u_0^2 + \frac{2e\phi}{m}.$$

but

$$n = \frac{-j}{eu} = -\frac{j}{e\sqrt{u_0^2 + \frac{2e\phi}{m}}}.$$

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Also

$$\nabla E = \frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{ne}{\epsilon_0} = \frac{j}{\epsilon_0\sqrt{u_0^2 + \frac{2e\phi}{m}}}.$$

Now we have 2 first oder differential equations:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = -E$$

$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{j}{\epsilon_0\sqrt{u_0^2 + \frac{2e\phi}{m}}} = -\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$

2 MAGNETEOHYDRODYNAMIC

2.1 *Parameters*

Mass density:

$$\rho \equiv n_e m_e + n_i m_i \approx n(n_e + n_i)$$

Fluid velocity:

$$\bar{\mathbf{u}} \equiv \frac{1}{\rho}(n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i) \approx \frac{m_e \mathbf{u}_e + m_i \mathbf{u}_i}{m_e + m_i}$$

Current density:

$$\mathbf{j} \equiv e(m_i \mathbf{u}_i - n_e \mathbf{u}_e) \approx n\, e(\mathbf{u}_i - \mathbf{u}_e)$$

2.2 *Continuity equation*

$$\frac{\partial n_i}{\partial t} + \nabla(n_i \mathbf{u_i}) = 0 \quad \Big| \cdot m_i$$

$$\frac{\partial n_e}{\partial t} + \nabla(n_e \mathbf{u_e}) = 0 \quad \Big| \cdot m_e$$

Add

$$\frac{\partial}{\partial t}\underbrace{(n_e m_e + n_i m_i)}_{\rho} + \nabla\underbrace{(n_e m_e \mathbf{u_e} + n_i m_i \mathbf{u_i})}_{\rho\bar{\mathbf{u}}} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0.}$$

2.3 *Momentum equation*

Momentum equations for electron and ion fluids:

$$\frac{\partial}{\partial t} m_e n_e \mathbf{u}_e + \underbrace{m_e n_e (\mathbf{u}_e \cdot \nabla)\mathbf{u}_e}_{\approx 0} = -e n_e m_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_e m_e \mathbf{g} - \nabla p_e + P_{ei} \quad (1)$$

$$\frac{\partial}{\partial t} m_i n_i \mathbf{u}_i + \underbrace{m_i n_i (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i}_{\approx 0} = e n_i m_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + n_i m_i \mathbf{g} - \nabla p_i + P_{ie} \quad (2)$$

Gravity is just a placeholder for any non-magnetic force. $P_{ei} = -P_{ie}$ describes the friction between the fluids. Add Eqs. (1) and (2) gives

$$n \frac{\partial}{\partial t} (m_i \mathbf{u}_i + m_e \mathbf{u}_e) = e n (\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B} - \nabla p + n(m_i + m_e)\mathbf{g}. \quad (3)$$

The electric field cancels due to quasi neutrality. The resulting MHD momentum equation is then

$$\boxed{\rho \frac{\partial \bar{\mathbf{u}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g}.}$$

We have lost any dependence on the friction term, which we need to recover. Obviously, we need another equation.

2.4 *Generalized Ohm's law*

We start again with the fluid momentum equations,

$$\frac{\partial}{\partial t} m_i n_i \mathbf{u}_i = + e n_i m_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + n_i m_i \mathbf{g} - \nabla p_i + P_{ie} \quad \Big| \cdot m_e$$

$$\frac{\partial}{\partial t} m_e n_e \mathbf{u}_e = - e n_e m_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_e m_e \mathbf{g} - \nabla p_e + P_{ei} \quad \Big| \cdot m_i$$

but this time we subtract them

$$\begin{aligned} m_i m_e n \frac{\partial}{\partial t} (\mathbf{u}_i - \mathbf{u}_e) =& e n (m_e + m_i)\mathbf{E} + e n (m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} \\ & - m_e \nabla p_i + m_i \nabla p_e - (m_e + m_i) P_{ei} \end{aligned}$$

We will derive an expression for P_{ie} later of this semester. For now we note that

$P_{ei} \sim$ Coulomb force $\sim e^2$

$P_{ei} \sim n_e$ and $n_i \sim n^2$

$P_{ei} \sim$ relative velocities $\sim (\mathbf{u}_i - \mathbf{u}_e)$

and thus

$$P_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e),$$

where the proportionality constant is the *resistivity*. Now

$$\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) = e\,\rho\mathbf{E} + e\,n(m_i\mathbf{u}_e + m_e\mathbf{u}_i)\times\mathbf{B}$$

$$- m_e\nabla p_i + m_i\nabla p_e - (m_e + m_i)\eta e\,n\,\mathbf{j}$$

Using that

$$m_e\mathbf{u}_i + m_i\mathbf{u}_e = m_i\mathbf{u}_i + m_e\mathbf{u}_e + m_i(\mathbf{u}_e - \mathbf{u}_i) + m_e(\mathbf{u}_i - \mathbf{u}_e)$$

$$= \frac{\rho}{n}\bar{\mathbf{u}} - (m_i - m_e)\frac{\mathbf{j}}{n\,e}$$

and hence

$$\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) = e\,\rho\mathbf{E} + e\,\rho\,\bar{\mathbf{u}}\times\mathbf{B} - (m_i - m_e)\mathbf{j}\times\mathbf{B}$$

$$- m_e\nabla p_i + m_i\nabla p_e - \rho\,e\,\eta\mathbf{j}$$

After dividing by ρe and rearranging terms

$$\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} - \eta\mathbf{j} = \frac{1}{e\rho}\left\{\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) + (m_i - m_e)\mathbf{j}\times\mathbf{B} + m_e\nabla p_i - m_i\nabla p_e\right\}$$

For MHD approximation we assume slow enough motions for $\frac{\partial}{\partial t}$ to be neglected. Slow enough means slower than ω_c^{-1}. We also take the limit $m_i \gg m_i$ and get the *generalized Ohm's law*

$$\boxed{\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{e\,n}\left(\mathbf{j}\times\mathbf{B} - \nabla p_e\right).}$$

For many plasma's the term in the brackets can be neglected

$$\boxed{\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} = \eta\mathbf{j}.}$$

The case of $\eta = 0$ is called *ideal MHD*.

2.5 *The Magnetohydrodynamics (MHD) equations*

Continuity equation:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{\mathbf{u}}) = 0 \tag{4}$$

Pressure:

$$pV^{\gamma} = \text{const.} \tag{5}$$

Momentum equation

$$\rho\frac{\partial\bar{\mathbf{u}}}{\partial t} = \mathbf{j}\times\mathbf{B} - \nabla p + \rho\mathbf{g} \tag{6}$$

Generalized Ohm's law

$$\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} = \eta\mathbf{j} \tag{7}$$

Faraday's law

$$\nabla\times\mathbf{E} = -\frac{\partial t}{\partial\mathbf{B}} \tag{8}$$

Ampere's law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} \tag{9}$$

Gauss' law

$$\nabla\cdot\mathbf{B} = 0 \tag{10}$$

3 CONVECTION AND DIFFUSION

In the previous section we derived the *generalized Ohm's law*. We now use Maxwell's equations to eliminate **j** from it

$$\mathbf{j} = \frac{1}{\eta}\left(\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B}\right) \quad \Big| \leftarrow \nabla\times\mathbf{B} = \mu_0\mathbf{j}$$

$$\nabla\times\mathbf{B} = \frac{\mu_0}{\eta}\left(\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B}\right) \quad \Big| \nabla\times$$

$$\nabla\times(\nabla\times\mathbf{B}) = \frac{\mu_0}{\eta}(\underbrace{\nabla\times\mathbf{E}}_{-\frac{\partial}{\partial t}\mathbf{B}} + \nabla\times(\bar{\mathbf{u}}\times\mathbf{B}))$$

$$\nabla(\underbrace{\nabla\cdot\mathbf{B}}_{0}) - \nabla^2\mathbf{B} = \frac{\mu_0}{\eta}(-\frac{\partial}{\partial t}\mathbf{B} + \nabla\times(\bar{\mathbf{u}}\times\mathbf{B}))$$

and find that

$$\boxed{\frac{\partial}{\partial t}\mathbf{B} = \underbrace{\nabla\times(\bar{\mathbf{u}}\times\mathbf{B})}_{\text{Convection}} + \underbrace{\frac{\eta}{\mu_0}}_{Diffusion}\nabla^2\mathbf{B}\,.} \tag{11}$$

3.1 *Magnetic diffusion*

If the so-called *convection term* $\nabla \times (\bar{\mathbf{u}} \times \mathbf{B})$ in Eq. (11) can be ignored, then this equation takes the form of a *Diffusion equation* for the magnetic field $\mathbf{B}$

$$\frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B}$$

with the magnetic diffusion coefficient

$$D_m = (\mu_0 \sigma)^{-1}$$

and the conductivity $\sigma = \eta^{-1}$. Let us now characterize the magnitude of the magnetic diffusion. The typical diffusion time scale τ_d can be found by replacing $\nabla^2 B$ by B/L_B^2, where L_B is the characteristic gradient of the magnetic field. Then

$$\frac{\partial B}{\partial t} \approx D_m B / L_B^2$$

or

$$\frac{B}{\dot{B}} \approx \frac{L_B^2}{D_m},$$

and hence

$$B \sim \exp\left(\pm t/\tau_d\right),$$

where

$$\boxed{\tau_d = \frac{L_B^2}{D_m} = \mu_0 \sigma L_B^2.} \tag{12}$$

is the diffusion time scale.

3.1.1 Diffusion in solar wind

Let us consider the solar wind flowing from the Sun to Earth at a typical speed of $v_{sw} = 500\,\mathrm{km/s}$, which takes about $\tau_{sw} \approx 3.5\,\mathrm{d}$. Because the solar wind plasma is cold and collisions with neutrals are rare, we only have to consider Coulomb collisions, which implies for Eq. (12) that

$$\tau_d \approx 0.3 L_B^2.$$

What is the typical length scale of the magnetic field diffusion during this period of time? Setting τ_d to the solar wind travel time τ_{sw} we find that

$$\tau_d = \tau_{sw} = 3.5\,\mathrm{d} \approx 0.3 L_B^2,$$

and thus

$$L_B \approx 1.9\sqrt{3.5\,\mathrm{d}} \approx 10^3\,\mathrm{m}.$$

The resulting diffusion length scale is much shorter than the distance between the Sun and Earth, which means that there is effectively no diffusion going on and the field is (practically) *frozen in* the solar wind plasma.

3.1.2 Diffusion in the Earth region

But the in the Earth region the situation is very different because of the prominence of collisions between the solar wind plasma particles with the neutrals of the Earth' atmosphere. Here $\sigma \sim 10^{-3}$ S/m and

$$\tau_d \approx 10^{-9} L_B^2.$$

Under such conditions, structures of tens of kilometers become diffusive within 1 second.

3.2 *Frozen-in magnetic flux*

If the conductivity approaches infinity or L_B becomes very large, Eq. (11) becomes

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}). \tag{13}$$

This equation is called the *hydromagnetic theorem* in analogy to the equation for the vorticity of non-viscous fluids. The theorem basically says that the field lines move with the plasma – the field appears to be *frozen-in the plasma*. The equation above is equivalent to

$$\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}),$$

and hence

$$0 = \nabla \times (\mathbf{E} + \bar{\mathbf{u}} \times \mathbf{B}).$$

The meaning of this identity is that the electric field disappears in frames co-moving with the plasma, or that electric fields can only result from Lorentz transformations.

3.3 *Magnetic Reynolds numbers*

We would like to have an easy means to verify whether the plasma is diffusive of convective. To this aim let us transform Eq. (11)

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}) + \frac{\eta}{\mu_0}\nabla^2 \mathbf{B}$$

into a dimensionless form

$$\frac{B}{\tau} = \frac{v \cdot B}{L_B} + \frac{B}{\tau_d}.$$

Here, v is the average plasma speed perpendicular to the field and L_B and τ_d are the characteristic length scale and diffusion time scale of the magnetic field, which we have introduced in 3.1.1. The ratio of the first and second term of the last expression

$$R_m = \frac{\frac{v \cdot B}{L_B}}{\frac{B}{\tau_d}} = \frac{v \cdot \tau_d}{L_B}$$

is the *magnetic Reynolds number*

$$\boxed{R_m = v L_B \mu_0 \sigma,}$$

which allows us to characterize the plasma state:

$$R_m \begin{array}{ll} \gg 1 & \text{flow dominated} \\ \ll 1 & \text{diffusion dominated.} \end{array}$$

We already mentioned that $R_m \gg 1$ implies that the magnetic field is frozen-in the plasma. To understand this strange claim better let us investigate the time dependence of the magnetic flux Φ encircled by a closed loop $c(t)$ around a bundle of frozen-in magnetic field lines:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \underbrace{\int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{A}}_{\text{expl. time dependence of } \mathbf{B}} + \underbrace{\int_C \mathbf{B} \cdot (\mathbf{u} \times d\mathbf{l})}_{\text{extra flux enclosed as curve moves}}$$

$$= \int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{A} + \int_C d\mathbf{l} \cdot (\mathbf{B} \times \mathbf{u}),$$

and after using Stoke's law

$$= \int_S \left\{ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u}\mathbf{B}\times) \right\}.$$

The hydromagnetic theorem (13) implies that $\{\ldots\} = 0$, which means that the magnetic flux encircled by a closed loop is conserved even if the loop is moving at a relative speed. Bundles of magnetic field lines frozen into the plasma are often called *flux tubes*.